

Week 6

Brenier's theorem: Let $X=Y=\mathbb{R}^d$, $c(x,y) = \frac{1}{2}|x-y|^2$, let $\mu, \nu \in \mathcal{P}(\mathbb{R}^d)$, with $\int |x|^2 d\mu + \int |y|^2 d\nu < +\infty$, $\mu \ll \mathcal{L}^d$.

Then, there exists a unique optimizer $\bar{\gamma}$ in (KP).

In addition, $\exists T = \nabla \varphi$, φ convex, s.t. $\bar{\gamma} = (\text{Id}, T)_\# \mu$,

$T = \nabla \varphi$ is the unique optimizer in (MP).

Corollary: Let c be continuous, bounded below. Then,

$\bar{\gamma}$ is optimal in (KP) $\Leftrightarrow \text{supp } \bar{\gamma}$ is c -cyclic monotone $\Leftrightarrow \exists \varphi$ c -convex s.t. $\text{supp } \bar{\gamma} \subseteq \partial^c \varphi$

Prop:

- $\varphi(x) + \varphi^c(y) + c(x,y) \geq 0 \quad \forall x \in X, y \in Y.$
- $\varphi(x) + \varphi^c(y) + c(x,y) = 0 \Leftrightarrow y \in \partial^c \varphi(x).$

Proof:

▷ Existence: By Br., $\text{supp } \bar{\gamma}$ is c -cyclic monotone, $\text{supp } \bar{\gamma} \subseteq \partial \varphi$, φ convex.

By Prop., $\varphi(x) + \varphi^*(y) = x \cdot y$ on $\text{supp } \bar{\gamma} \subseteq \partial \varphi$

$\Rightarrow (\varphi(x), \varphi^*(y))$ are finite for $\bar{\gamma}$ -a.e. $(x,y) \Rightarrow \varphi(x)$ is finite for μ -a.e. $x \rightarrow$

$\rightarrow \varphi$ is μ -a.e. differentiable, $\mu \ll \mathcal{L}^d$. (convex functions are diff. a.e. where they are finite.)

Let A with $\mu(A) = 0$ s.t. $\nabla \varphi(x)$ exists $\forall x \in \mathbb{R}^d \setminus A$. We claim that

$y = \nabla \varphi(x)$ for $\bar{\gamma}$ -a.e. (x,y) (*).

Let us prove first that $y = \nabla \varphi(x) \quad \forall (x,y) \in ((\mathbb{R}^d \setminus A) \times \mathbb{R}^d) \cap \text{supp } \bar{\gamma}$.

This will be enough to show (*), since $\bar{\gamma}(A \times \mathbb{R}^d) = \mu(A) = 0$.

We have $(x,y) \in \partial \varphi$, and φ differentiable at x , so $y = \nabla \varphi(x)$,

and $(x,y) = (x, \nabla \varphi(x))$ $\bar{\gamma}$ -a.e.

We now claim that $\bar{\gamma} = (\text{Id}, \nabla \varphi)_\# \mu$. Indeed, $\forall F$ measurable, $F \geq 0$,

$$\int F(x,y) d\bar{\gamma}(x,y) = \int F(x, \nabla \varphi(x)) d\bar{\gamma} = \int F(x, \nabla \varphi(x)) d\mu(x) = \int F(x,y) d[(\text{Id}, \nabla \varphi)_\# \mu] \quad \checkmark$$

▷ Uniqueness: By contradiction, $\exists \gamma_1, \gamma_2$ optimal in (KP)

$\Rightarrow \frac{\gamma_1 + \gamma_2}{2}$ is also optimal. Indeed:

$$\text{Cost} \left(\frac{\gamma_1 + \gamma_2}{2} \right) = \int c \, d \left(\frac{\gamma_1 + \gamma_2}{2} \right) = \frac{1}{2} \int c \, d\gamma_1 + \frac{1}{2} \int c \, d\gamma_2 = (KP)$$

$\frac{\gamma_1 + \gamma_2}{2}$ has the same marginals as γ_1 & γ_2 .

By the first part of the proof, $\frac{\gamma_1 + \gamma_2}{2} = (\text{Id}, \nabla \varphi)_{\#} \mu$, and even more precisely, $y = \nabla \varphi(x) \quad \forall (x, y) \in \text{supp} \frac{\gamma_1 + \gamma_2}{2} \cap ((\mathbb{R}^d \setminus A) \times \mathbb{R}^d)$.

$$\subseteq \text{supp} \gamma_1 \cup \text{supp} \gamma_2.$$

In particular, $y = \nabla \varphi(x) \quad \forall (x, y) \in \text{supp} \gamma_i \cap [(\mathbb{R}^d \setminus A) \times \mathbb{R}^d] \quad i=1,2$.

As before:

$$\int F(x, y) \, d\gamma_1 = \int F(x, \nabla \varphi(x)) \, d\gamma_1 = \int F(x, y) \, d((\text{Id}, \nabla \varphi)_{\#} \mu) \Rightarrow$$

$$\gamma_1 = (\text{Id}, \nabla \varphi)_{\#} \mu = \gamma_2. \quad \text{Contradiction.}$$

Relation with Monge: Let φ be the convex function constructed in the existence part. We know in general:

$$(KP) \leq (MP) \leq \int | \nabla \varphi(x) - x |^2 \, d\mu(x)$$

③ $\rightarrow$ ||

$\rightarrow \nabla \varphi$ is a competitor in MP.

$$\int |x - y|^2 \, d((\text{Id}, \nabla \varphi)_{\#} \mu) = \int |x - \nabla \varphi(x)|^2 \, d\mu(x)$$

Hence, they are all equal. (Uniqueness of T in (MP) follows. $\square$)

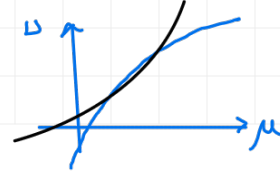
Corollary: Let $\mu, \nu \ll \mathbb{R}^d$, $\int |x|^2 \, d\mu + \int |y|^2 \, d\nu < +\infty$.

Let $\nabla \varphi$ optimal in (MP) with marginals μ and ν ,
and $\nabla \psi$ optimal in (MP) with marginals ν and μ .

Then, $\nabla \varphi$ is μ -a.e. invertible and its inverse is $\nabla \psi$:

$$\nabla \psi \circ \nabla \varphi = \text{Id} \quad \mu\text{-a.e.}, \quad \nabla \varphi \circ \nabla \psi = \text{Id} \quad \nu\text{-a.e.}$$

Idea: Use uniqueness in (KP) and symmetry:



Proof: $\int |\nabla \varphi(x) - x|^2 d\mu(x) = \int |x-y|^2 d(\text{Id}, \nabla \varphi)_\# \mu = (KP) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int |x-y|^2 d\gamma$
 $\int |\nabla \psi(y) - x|^2 d\nu(y) = \int |x-y|^2 d(\nabla \psi, \text{Id})_\# \nu = (KP)$

By uniqueness of optimizer in (KP) , $(\text{Id}, \nabla \varphi)_\# \mu = (\nabla \psi, \text{Id})_\# \nu$ (*)

We claim that $\nabla \psi \circ \nabla \varphi - x = 0$ for μ -a.e. x . Indeed:

$$\int |\nabla \psi \circ \nabla \varphi(x) - x|^2 d\mu = \int |\nabla \psi(y) - x|^2 d[(\text{Id}, \nabla \varphi)_\# \mu] \stackrel{(*)}{=} \int |\nabla \psi(y) - x|^2 d(\nabla \psi, \text{Id})_\# \nu = \int |\nabla \psi(y) - \nabla \psi(y)|^2 d\nu = 0 \quad \square$$

Theorem (Brenier's theorem for general costs):

Let $X = Y = \mathbb{R}^d$, (X Riem. manifold, Y metric space would be enough)

$\mu \ll \mathbb{R}^d$, $\text{supp } \nu$ compact. Let $c \in C^\infty(X \times Y)$ bounded below,

and assume $(KP) = \inf_{\gamma \in \Gamma(\mu, \nu)} \int c(x,y) d\gamma < +\infty$. Assume also:

- $\forall y \in \text{supp } \nu$, $\mathbb{R}^d \ni x \mapsto c(x,y)$ is differentiable.
- $\forall y \in \text{supp } \nu$, $R > 0$, $|\nabla_x c(x,y)| \leq C_R$ for a.e. $x \in B_R$.
- $\forall x \in \mathbb{R}^d$, $\mathbb{R}^d \ni y \mapsto \nabla_x c(x,y) \in \mathbb{R}^d$ is injective $\leftarrow$ TWIST CONDITION

Then, there exists a unique optimizer $\bar{\gamma}$ in (KP) , with $\bar{\gamma} = (\text{Id}, T)_\# \mu$,

and for some c -convex function φ , $\nabla_x c(x, T(x)) + \nabla \varphi = 0$.

Proof: Let $\bar{\gamma}$ optimal in (KP) . By Cor., $\exists \varphi$ c -convex s.t. $\text{supp } \bar{\gamma} \subseteq \partial^c \varphi$.

By Prop.,

$$\varphi(x) + \varphi^c(y) + c(x,y) \stackrel{\geq 0 \text{ always}}{=} 0 \quad \forall (x,y) \in \text{supp } \bar{\gamma} \subseteq \partial^c \varphi \quad (*).$$

We would like φ to be locally Lipschitz. We replace φ by $\tilde{\varphi}$ in

the previous formula, with $\tilde{\varphi}$ locally Lipschitz:

$$\text{Let } \tilde{\varphi}(x) := \sup_{y \in \text{supp } \nu} \{ \underbrace{-c(x,y) - \varphi^c(y)}_{\leq \varphi(x)} \} \rightarrow \text{extended to } -\infty \text{ outside } \text{supp } \nu$$

Since $\text{supp } \bar{\gamma} \subseteq \mathbb{R}^d \times \text{supp } \nu$ and $(*)$:

$$0 \leq \tilde{\varphi}(x) + \varphi^c(y) + c(x,y) \leq \varphi(x) + \varphi^c(y) + c(x,y) = 0 \quad \forall (x,y) \in \text{supp } \bar{\gamma}.$$

$\tilde{\varphi}$ is loc. Lipschitz because $x \mapsto -c(x,y) - \varphi^c(y)$ are C_R -Lipschitz in B_R , $R > 0$, by 1st and 2nd assumptions.

Hence, $\sup_y \{-c(x,y) - \varphi^c(y)\}$ is C_α -Lipschitz. By Rademacher's theorem, $\tilde{\varphi}$ is μ -a.e. differentiable ($\mu \ll \mathbb{R}^d$)

Let A be the set of non-differentiability, $\mu(A) = 0$.

Let $(x,y) \in \text{supp } \tilde{\gamma}$, $x \notin A$. Then, $z \mapsto \tilde{\varphi}(z) + \varphi^c(y) + c(z,y)$ has a min at $z=x$, and it is differentiable there $\Rightarrow \nabla \tilde{\varphi}(x) + \nabla_x c(x,y) = 0$.

Since, by assumption, $\nabla_x c(x,y)$ is injective in y , the equation has at most one solution, $y = T(x)$. As in the other proof (with quadratic cost) we showed:

$$\text{supp } \tilde{\gamma} \cap [(\mathbb{R}^d \setminus A) \times \text{supp } \nu] \subseteq \text{graph } T,$$

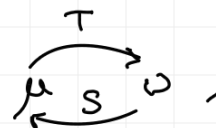
we conclude $\tilde{\gamma} = (\text{Id}, T)_\# \mu$.

• The uniqueness works as in the other proof. $\square$

Remark on the proof: • Twist was used to invert the optimality condition.

• $\|\nabla_x c\|_{L^\infty}$ was used to get $\tilde{\varphi}$ Lipschitz.

• if $\mu, \nu \ll \mathbb{R}^d$ and T, S are optimal maps, then S and T are inverse of each other.



Remark: $c(x,y) = |x-y|^p$, $p \geq 1$, when does Brenier's theorem hold?